The inverse of $I - ba$ and $I - ab$

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1 Relationship between the inverse of $I - ba$ and $I - ab$

In a Monoid,There is a relationship between the inverse of $I - ba$ and $I - ab$.Thus, $I - ba$ is invertible iff $I - ab$ and we have

$$(I - ba)^{-1} = I + b(I - ab)^{-1}a$$

we can check the formula by simply compute it.

But how can we find this formula ?

unformally, we have

$$(I - ba)^{-1} = I + \sum_{n=1}^{\infty}(ba)^n = I + b(I + \sum_{n=1}^{\infty}(ab)^n a = I + b(I - ab)^{-1}a$$

So we generate our formula unstrictly,then prove it strictly.

2 The Sherman-Morrison-Woodbury Formula

The Sherman-Morrison-Woodbury formula gives a convenient expression for the inverse of the matrix $A + UV^T$ where $A \in \mathbb{R}^{n\times n}$ and U and V are $n \times k$.

$$(A + UV^T)^{-1} = A^{-1} + A^{-1}U(I + V^T A^{-1})^{-1}V^T A^{-1}$$

It can be proved by using $(I - ba)^{-1} = I + b(I - ab)^{-1}a$

The k= 1 case is particularly useful. If $A \in \mathbb{R}^{n\times n}$ is nonsingular, $u, v \in \mathbb{R}^n$ and $\alpha = 1 - v^T A^{-1}u \neq 0$,then

$$(A + uv^T)^{-1} = A^{-1} - \frac{1}{\alpha}A^{-1}uv^T A^{-1}$$

3 There is a problem

In $1 we have,we assume that, the formula is generated in a monoid, but the vector and matrix didn't form a monoid since a AU is undefined if A is a $n \times n$ matrix,but U is a $k \times n$ matrix. how can be define a Algebra structrual for this useful struct ?