# Some results about group

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March 18, 2016

I found some results really interesting about group which listed at the book *Basic Algebra by NATHAN JACOBSON*

## 1 Invertiblity in a monoid

Let M donates a monoid,any of the following conditions

1. $ab = ca = 1$
2. $aba = a, ab^2a = 1$

can conclude that $a$ is invertible with b as inverse.

1. *proof:* if $ab = ca = 1$ then,

   $$b = 1 \cdot b = cab = c \cdot 1 = c$$

   thus

   $$ab = ba = 1$$

2. if $aba = a, ab^2a = 1$ then,

   $$1 = ab^2a = abab^2a = ab$$

   $$1 = ab^2a = abba = ba$$

   end the proof.

## 2 Semigroup with some properities

Let $G$ be a semigroup having following properites

**(a)** $G$ contains right unit $1_r$,thus $\forall a \in G, a1_r = a$

**(b)** $\forall a \in G$ have a right inverse to $1_r$ $(ab = 1_r)$.

then $G$ is a group.
*proof:* $\forall a \in G$ we have $b \in G$ such thar

$$ab = 1_r$$

then there exists $c \in G$ that

$$bc = 1_r$$

hence we have

$$a = a1_r = abc = 1_r c = 1_r 1_r c = 1_r a$$

so we have $\forall a \in G$

$$a1_r = 1_r a = a$$

thus $1_r$ is the unit in $G$ and G is a monoid.Using conclusion in previous chapter,we know that $a$ is invertible for all $a \in G$.So G is a group.

## 3 Semigroup with solvable system

$G$ is a monoid,if the equation $ax = b$ and $ya = b$ is solvable for any $a, b \in G$ then $G$ is a group.
*proof:*
we random choose $a_0 \in G$ and we have $x_0$ such that

$$a_0 x_0 = a_0$$

$\forall a \in G$ we have

$$a = ya_0 = ya_0 x_0 = ax_0$$

and $\forall a \in G$ there exist a $b$ such that

$$ab = x_0$$

. end our prove by foregoing conclusion.

# 4 Finite semigroup with cancellation law

Let $G$ donate a finite semigroup, is G satisfies

$$ax = ay \Rightarrow x = y$$

and

$$xa = ya \Rightarrow x = y$$

G is a group.
*proof:* $\forall a \in G$ we consider a map $L_a\colon G \to G$

$$L_a(g) = ag \text{ for all } g \in G$$

since $ax = ay \Rightarrow x = y$ we know that $L_a$ is injective, $|G|$ is finite,we have $L_a$ is bijective.
Similiarly,we can define a bijective map $R_a$.
Thus $G$ is a Semigroup with solvable system which refer to be a group by previos chapter.