Relationship between Spectral Radius, Numerical Radius and Spectral Norm

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1 Definition

1.1 Spectral Radius $\rho(A)$

$\rho(A)$ is defined as follows:

$$\rho(A) = \max_k |\lambda_k|$$

where λ_k donates the k-th eigenvaule of matrix A.

1.2 Numerical Radius $w(A)$

$$w(A) = \max_{\|x\|_2=1} |x^* Ax|$$

We can check that $w(A)$ is a norm on $\mathbb{C}^{n\times n}$,but not a matrix norm on $\mathbb{C}^{n\times n}$.

1. $w(A) \geq 0$ is trival to prove, if $w(A) = 0$,we know for all $x \in \mathbb{C}^n$

$$x^* Ax = 0 \Longrightarrow x^* A^* x = 0 \Longrightarrow x^* \frac{A + A^*}{2} x = 0 \text{ and } x^* \frac{A - A^*}{2} x = 0$$

   Since $\frac{A+A^*}{2}$ and $\frac{A-A^*}{2}$ are normal matrix,we know that

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2} = 0 + 0 = 0$$

2. $w(aA) = |a|w(A)$ and $w(A + B) \leq w(A) + w(B)$ is apparently correct.

3. We,however,don't have $w(AB) \leq w(A)w(B)$.For instance,let $B = A^*$ where

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so we have

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$w(AB) = 1$ and $w(A) = w(B) = 1/2$ conflict to $w(AB) \leq w(A)w(B)$

So $w(A)$ is a norm but not a matrix norm on $\mathbb{C}^n$.

1.3 Spectral norm $\|A\|_2$

We now define $\|A\|_2$ as

$$\|A\|_2 = \max_{0 \neq x \in \mathbb{C}^n} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2=1} \|Ax\|_2 = \max_{\|x\|_2=1} \sqrt{|x^* A^* A x|} = \sqrt{\lambda_{\max}(A^* A)} = \sigma_{\max}(A)$$

We can easy check that $\|A\|_2$ is a matrix norm.

2 $\rho(A) \leq w(A) \leq \|A\|_2$

let λ be a eigenvaule of A and x is its correspond unit eigenvector,thus

$$Ax = \lambda x$$

so

$$x^* A x = \lambda x^* x = \lambda \|x\|_2 = \lambda$$

hence

$$|\lambda| = |x^* A x| \leq \max_{\|x\|_2=1} |x^* A x| \leq w(A)$$

proved that $\rho(A) \leq w(A)$

Since

$$w^2(A) = \max_{\|x\|_2=1} x^* A x x^* A^* x$$

and

$$Ax = U^* \begin{pmatrix} \|Ax\|_2 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

so

$$x^*Axx^*A^*x = x^*U^* \begin{pmatrix} \|Ax\|_2^2 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} Ux \leq \|A\|_2^2$$

hence $w(A) \leq \|A\|_2$ end our prove.

In addition, if A is a normal matrix $\rho(A) = w(A) = \|A\|_2$ can be easy proved using Schur decomposition.

3 $\frac{1}{2}\|A\|_2 \leq w(A) \leq \|A\|_2$

We only need to prove $\|A\|_2 \leq 2w(A)$

Since $(x^*A^*x)^* = x^*A^*x$ So

$$\mathrm{Re}(x^*Ax) = x^*\frac{A+A^*}{2}x$$

and

$$\mathrm{Im}(x^*Ax) = x^*\frac{A-A^*}{2i}x$$

Hence

$$\begin{aligned} \|A\|_2 =& \|\frac{A+A^*}{2} + \frac{A-A^*}{2}\|_2 \leq \|\frac{A+A^*}{2}\|_2 + \|\frac{A-A^*}{2}\|_2 \\ =& w(\frac{A+A^*}{2}) + w(\frac{A-A^*}{2}) \leq w(A) + w(A) = 2w(A) \end{aligned}$$

end our prove.

In additon,the equation can be achieve with above example in this paper.

At last, we expound that $\rho(A)$,$w(A)$,$\|A\|_2$ are all unitary invariance. The prove is trival.