Some problems on First Class

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I only considered some of the problems. I spent about 25 hours to consider these problems and learn TEX. I choose to use English because It is real fussy to use Chinese in TEX.

1 Complex Symmetric Matrix

Define: A is called *Complex Symmetirc Matrix*
if it satisfies that

$$A = A^T$$

where A is a complex matrix.

None of property did I find in *Complex Symmetirc Matrix*, since A isn't a *normal matrix*($A^*A = AA^*$)in general,so it can't be diaged by a unitary matrix.
Here is a example:

$$A = \begin{pmatrix} 0 & i & 0 \\ i & 1 & 2i \\ 0 & 2i & 2 \end{pmatrix}$$

We can MATLAB to illustate that A is can't be diaged by a unitary matrix, and their eigenvalues are all complex.

Since no good property found in complex symmetric matrix, we usually consider a Hermite matrix instead of complex symmetric matrix. What's more, The A' in MATLAB donates the conjugate transpose of A instead of transpose of A. Real symmetric matrix which we usually considered, essentially, is a kind of Hermite matrix.

2 A Matrix Factorization

We all know that any complex square matrix A,there are matrix B and C such that

$$A = B + C$$

where B is Herimite matrix,and C is a anti-Hermite matrix. This factorization of A is unique.
Prove: suggest

$$A = B + C$$

holds,then we have

$$A^* = B^* + C^* = B - C$$

so

$$B = \frac{A + A^*}{2}, \ \ C = \frac{A - A^*}{2}$$

hence B and C are unique,and we can check that B is Herimite matrix and C is anti-Herimite matrix.

Similarly,we have

$$A = B + iC$$

where

$$B = \frac{A + A^*}{2}, \ \ C = \frac{A - A^*}{2i}$$

both B and C are unique Hermite matrix.

Matrix Inequality

$$|Re(\lambda(A))| \leq \rho(\frac{A + A^*}{2}) \leq \sigma_{max}(A)$$

$$|Im(\lambda(A))| \leq \rho(\frac{A - A^*}{2i}) \leq \sigma_{max}(A)$$

here $\lambda(A)$ represent any eigenvalue of square matrix A.

In order to prove the inequality briefly,we introduce a matrix norm $w(A)$ defined as follows,

$$w(A) \equiv \max_{\|x\|_2 = 1} |x^* A x| = w(A^*)$$

we can easy prove that $w(A)$ is a matrix norm and unitary invariant.

We have following inequality (it's well known and easy to prove)

$$\rho(A) \leq w(A) \leq \sigma_{\max}(A)$$

Equation holds for any *normal* matrix A.

Now, I'm going to prove our matrix inequality.
Let λ_0 be a eigenvalue of A, and x_0 be it's correspond eigenvector, we may wish to set that $\|x_0\|_2 = 1$. Thus we have

$$Ax_0 = \lambda_0 x_0$$

so

$$x_0^* Ax_0 = x_0^* \lambda_0 x_0 = \lambda_0 x_0^* x_0 = \lambda_0$$

hence we have,

$$x_0^* A^* x_0 = \overline{\lambda_0}$$

thus,

$$x_0^* \frac{A^* + A}{2} x_0 = \frac{\lambda_0 + \overline{\lambda_0}}{2} = Re\lambda_0$$

Note that $\frac{A^*+A}{2}$ is a hermite matrix, hence we have

$$|Re\lambda_0| = |x_0^* \frac{A^* + A}{2} x_0| \leq w(\frac{A^* + A}{2}) = \rho(\frac{A^* + A}{2})$$

On the other hand,we have

$$\rho(\frac{A^* + A}{2}) = w(\frac{A^* + A}{2}) \leq w(\frac{A^*}{2}) + w(\frac{A}{2}) = w(A) \leq \sigma(A)$$

Similarly,we can prove the second inequation.

The form of the inequation is real beautiful, They say that any eigenvalue of a square matrix A can be bounded by two Hermite matrix B, C where $A = B + iC$,and B, C is unique determined by A.

I explain my inequation in another way
$\forall A \in \mathrm{C}^{n \times n}$ there is a unique Hermite matrix B and anti-Hermite matrix C such that $A = B + iC$. In addition,the real part of $\lambda(A)$ is bounded by $\rho(B)$ and the image part of $\lambda(A)$ is bounded by $\rho(C)$.

Real Matrix Inequality

Since real symmetric matrix is a kind of Hermite matrix and $\rho(iC) = \rho(C)$, we can get the real form of our inequation as follows,

$$|Re(\lambda(A))| \leq \rho(\frac{A + A^T}{2}) \leq \sigma_{max}(A)$$

$$|Im(\lambda(A))| \leq \rho(\frac{A - A^T}{2}) \leq \sigma_{max}(A)$$

Inparticular,if A is a non-negetive matrix.Using theorem of Perron-Frobenius about non-negtive matrix, we know that A has a non-negtive eigenvalue and equal to $\rho(A)$, so using our inequality we have,

$$\rho(A) \leq \rho(\frac{A+A^T}{2}) \leq \sigma_{max}(A)$$

holds for any non-negtive square matrix.

Thanks to my teacher Mr.Wei who let me consider the relationship between $A = B+C$. I had no thought about this at first, but I soon found these interesting inequation when I choose some rand matrix to do experiment in MATLAB.I don't know whether the result had been found before, anyway, I'm so happy to discover and prove the inequation independently.

3 Minimax Principle

(Minimax principle) If ϕ: $\mathrm{C}^n \to \mathrm{R}$ is continue in C^n, then

$$s_k \equiv \max_{\substack{S \subseteq \mathrm{C}^n \\ \dim S = k}} \min_{\substack{x \in S \\ \|x\| = 1}} \phi(x) \leq \min_{\substack{T \subseteq \mathrm{C}^n \\ \dim T = n-k+1}} \max_{\substack{x \in T \\ \|x\| = 1}} \phi(x) \equiv t_k$$

Prove: There exists a $S_0 \subseteq \mathrm{C}^n$ such that

$$s_k = \min_{\substack{x \in S_0 \\ \|x\| = 1}} \phi(x)$$

$\forall T \subseteq \mathrm{C}^n, \dim T = n-k+1$,we have

$$\dim T \cap S_0 = \dim T + \dim S_0 - \dim T \cup S_0 \geq 1$$

thus we have $x_0 \in T \cap S_0, \|x_0\| = 1$.

$$\max_{\substack{x \in T \\ \|x\| = 1}} \phi(x) \geq \phi(x_0) \geq \min_{\substack{x \in S_0 \\ \|x\| = 1}} \phi(x) = s_k$$

hence

$$\max_{\substack{S \subseteq \mathrm{C}^n \\ \dim S = k}} \min_{\substack{x \in S \\ \|x\| = 1}} \phi(x) \leq \min_{\substack{T \subseteq \mathrm{C}^n \\ \dim T = n-k+1}} \max_{\substack{x \in T \\ \|x\| = 1}} \phi(x)$$

I try to prove that the opposite side is also true, but fail since the opposite side doesn't hold in general.

if A is Hermitian,then $\phi(x) = x^*Ax \in \mathrm{R}$,we choose $\|\cdot\| = \|\cdot\|_2$.Then we have $s_k = t_k = \lambda_k$. This is called **Courant-Fischer's minimax theorem**.

Since $\forall x \in \mathrm{C}^n, \phi(x) = x^*Ax \in \mathrm{R} \iff A$ is a Hermite matrix.
In order to generalize *Courant-Fischer's minimax theorem*,we choose vector *norm* to be

$$\|\cdot\| = \|\cdot\|_2$$

and from now on, define $\phi(x)$ and as follows,

$$\phi(x) = |x^*Ax|$$

If we arrange eigenvalues in absolute descending order.I'm going to prove that for any *normal* matrix A

$$t_k \equiv \min_{\substack{T \subseteq \mathrm{C}^n \\ \dim T = n-k+1}} \max_{\substack{x \in T \\ \|x\|_2 = 1}} |x^*Ax| = |\lambda_k|$$

Since A is a *normal* matrix,by Schur's theorem,we know A can be diaged by unitary matrix,that is to say, A have orthogonal eigenvectors $x_1, \cdots x_n$ coincide with it's eigenvalues $\lambda_1, \ldots \lambda_n$.
Let $T_0 = \sigma(x_k, \ldots x_n)$,then $\forall x \in T_0$

$$x = \langle x, x_k \rangle x_k +, \cdots + \langle x, x_n \rangle x_n$$

hence

$$|x^*Ax| = |\sum_{i=k}^{n} |\langle x, x_i \rangle|^2 x_i| \leq \sum_{i=k}^{n} |\langle x, \lambda_i \rangle|^2 |\lambda_i| \leq |\lambda_k| \sum_{i=k}^{n} |\langle x, x_i \rangle|^2 = |\lambda_k| \|x\|_2$$

so

$$t_k \leq \max_{\substack{x \in T_0 \\ \|x\|_2 = 1}} |x^*Ax| \leq |\lambda_k|$$

end my prove.

I'm now give an example shows that $t_k < |\lambda_k|$ maybe true for some k even the matrix is Hermitian.

Example 1.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$|\lambda_1| = |\lambda_2| = 1$ but $t_1 = 1, t_2 = 0$,thus $t_2 < |\lambda_2|$

Since A is anti-Hermitian $\iff$ iA is Hermitian. we can get coincide *Courant-Fischers minimax theorem* for anti-Hermite matrix.

$$i\lambda_i(A) = \max_{\substack{S \subseteq \mathrm{C}^n \\ \dim S = k}} \min_{\substack{x \in S \\ \|x\|_2 = 1}} x^* iAx = \min_{\substack{T \subseteq \mathrm{C}^n \\ \dim T = n-k+1}} \max_{\substack{x \in T \\ \|x\|_2 = 1}} x^* iAx$$

for any anti-Hermite matrix A.

4 Some Prove in Our Books

4.1 Brief prove of some theorem

1. **Courant-Fischer's minimax theorem** can be proved using dimension formula and spectral theory.

2. **Cauchy's interlace theorem** can be proved using **Courant-Fischer's minimax theorem**. so is **Weyl's theory**.

4.2 Shorter prove of some theorem

The prove of following result in our book(Page 5-6)is real fussy.

1. For any matrix A

$$\|A\|_2 = \sigma_{\max}(A)$$

Prove:

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \max_{\|x\|_2=1} [x^*(A^*A)x]^{\frac{1}{2}} = [\rho(A^*A)]^{\frac{1}{2}} = \sigma_{\max}(A)$$

2. For any *unitary* matrices Q and Z

$$\|A\|_F = \|QAZ\|_F$$

Prove:

$$\|QAZ\|_F = [tr(Z^*A^*Q^*QAZ)]^{\frac{1}{2}} = [tr(Z^*A^*AZ)]^{\frac{1}{2}} = [tr(ZZ^*A^*A)]^{\frac{1}{2}} = tr(A^*A)]^{\frac{1}{2}} = \|A\|_F$$

5 Condition Number

Let $\|\cdot\|$ be any matrix norm and A be an invertible matrix.The condition number of A is define as follows,

$$\kappa(A) \equiv \|A\| \cdot \|A^{-1}\|$$

but,what is the condition number for singular matrix ? Actually,we can replace A^{-1} to *Moore-Penrose's generalized inverse* A^{+} which satisfied following equations

$$AXA = A$$

$$XAX = X$$

$$(AX)^{*} = AX$$

$$(XA)^{*} = XA$$

we can prove that X is exists and unique,we mark it as A^{+}.
In fact,if $A = U\Delta V$ is a SVD of A,where $\Delta = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k, 0, \ldots, 0)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > 0$.
Then we have

$$A^{+} = V^{*}\hat{\Delta}U^{*}$$

where $\hat{\Delta} = \mathrm{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_k}, 0, \ldots, 0)$. In addition,if we have a norm for any matrix(not need to be square),then using *Moore-Penrose's generalized inverse* we can define condition number for any matrix as follows,

$$\kappa(A) \equiv \|A\| \cdot \|A^{+}\|$$

In particular,if A is a square matrix and $\|\cdot\| \equiv \|\cdot\|_2$.then

$$\kappa(A) = \|A\|_2 \cdot \|A^{+}\|_2 = \frac{\sigma_1(A)}{\sigma_k(A)}$$

here $\sigma_1(A)$ is greatest singular value of A, and $\sigma_k(A)$ is the leastest non-zero singular value of A.

6 I can't find such Norm

$\forall \epsilon > 0, \exists \|\cdot\|_{*}$ satisfies that

$$\|A\|_{*} < \rho(A) + \epsilon$$

hold for all matrix $A \in \mathrm{C}^{n \times n}$.

If such proposition is ture,I choose a $\epsilon_0 > 0$ then $\exists \| \cdot \|_*$ satisfies that

$$\|A\|_* < \rho(A) + \epsilon$$

hold for all matrix $A \in \mathrm{C}^{n\times n}$.
There must be a matrix A_1 in $\mathrm{C}^{n\times n}$ such that

$$\rho(A_1) < \|A_1\|_* < \|A_1\|_* + \epsilon_0$$

so

$$\|A_1\|_* = \rho(A_1) + \epsilon_1$$

where $0 < \epsilon_1 < \epsilon_0$ so

$$\|\frac{2\epsilon_0}{\epsilon_1} A_1\|_* = \frac{2\epsilon_0}{\epsilon_1}\|A_1\|_* = \frac{2\epsilon_0}{\epsilon_1}\rho(A_1) + \frac{2\epsilon_0}{\epsilon_1}\epsilon_1 = \rho(\frac{2\epsilon_0}{\epsilon_1} A_1) + 2\epsilon_0$$

contradicte the condition.

The following proposition may be right,but I couldn't prove it. $\forall A \in \mathrm{C}^{n\times n}, \forall \epsilon > 0, \exists \| \cdot \|_*$ satisfies that

$$\|A\|_* < \rho(A) + \epsilon$$