Mathematical Entertainments

Steven H. Weintraub*

For the general philosophy of this section, see Volume 9, No. 1 (1987). An asterisk (*) placed beside a problem indicates a submission without solution; a dagger (†) indicates that it is not new. Solutions should be received by November 1, 1987. Note that the column editor's address will change as of September 1. Both the current and new addresses are to be found below.

Problems

To coin a phrase: Competition 87-7 by the Column Editor

A famous apocryphal review states that a paper "fills a much-needed gap in the literature," and the column editor has seen a Ph.D. thesis in which the student thanks his advisor, whose help "cannot be underestimated." Readers are invited to submit additional bon mots along these lines.

Three squares: Problem 87-8 by M. Theodore (Sicily)

Find positive integers x, y, and z such that

$$x + y + z + x^2,$$
$$x + y + z + x^2 + y^2, \text{ and}$$
$$x + y + z + x^2 + y^2 + z^2$$

are all perfect squares.

* **Column editor's address (until August 31, 1987):** SFB 170, Mathematisches Institut, Bunsenstrasse 3-5, D-3400 Göttingen, Fed. Rep. Germany

(after September 1, 1987): Mathematisches Institut, Universität Bayreuth, Postfach 3008, D-8580 Bayreuth, Fed. Rep. Germany

Integer roots: Problem 87-9* by Frank Schmidt (Bryn Mawr College, USA)

For each integer $n > 2$, does there exist a polynomial P_n of degree n such that P_n and all its derivatives have integer roots, and such that

a) P_n has at least 3 distinct roots
or b) P_n has n distinct roots?

Editorial Note: *Several solvers suggested that the existence of such a simple solution means that the original problem should have been termed a "quickie," and that it should be re-posed in such a way that this solution is excluded. Their advice has been followed, and this problem has reappeared as problem 87-9 above.*

Solutions

86-2* by Jerrold W. Grossman (Oakland University, USA)

Determine all real x such that the following recursively defined sequence converges:

$$A_{n+2} = A_n/(1 + A_{n+1}),\ n \geq 0;$$
$$A_0 = 1,\ A_1 = x.$$

Solution by A. J. E. M. Janssen and D. L. A. Tjaden (Philips Research Laboratories, Eindhoven, Netherlands)

We rewrite the sequence slightly for convenience:

$$A_0(x) = 1,\ A_1(x) = x,$$
$$A_{n+2}(x) = A_n(x)/(1 + A_{n+1}(x)) \text{ for } n \geqslant 0. \quad \textbf{(1)}$$

We will show that there is exactly one value of x, $x_0 = 0.73733830336929\ldots$, for which the sequence converges, in which case it is monotone decreasing with limit 0. The proof proceeds in several steps.

1. $A_n(x) > 0$, $x > 0$, $n = 0, 1, \ldots$, and $A_n(0) = 1$ for n even, 0 for n odd.

Proof. Immediate from (1).

2. If $\lim_{n\to\infty} A_n(x)$ exists, it is 0.

Proof. Let

$$a(x) = \lim_{n\to\infty} A_{2n}(x),\ b(x) = \lim_{n\to\infty} A_{2n+1}(x). \quad \textbf{(2)}$$

Then it follows from (1) that

$$a(x) = a(x)/(1 + b(x)),\ b(x) = b(x)/(1 + a(x)); \quad \textbf{(3)}$$

i.e., $a(x) = 0$ or $b(x) = 0$.

We will show that for $x \geqslant 0$, $a(x)$ and $b(x)$ are continuous, and that there is exactly one $x \geqslant 0$ for which $a(x) = b(x) = 0$, and x_0 is this value.

3. $\lim_{n\to\infty} A_n(x)$ does not exist for $x \leqslant 0$.

Proof. Suppose that $\lim_{n\to\infty} A_n(x) = 0$. Since

$$A_{2n+1}(x) = A_{2n-1}(x)/(1 + A_{2n}(x)), \quad \textbf{(4)}$$
$$A_{2n}(x) = A_{2n-2}(x)/(1 + A_{2n-1}(x)), \quad \textbf{(5)}$$

we see that $A_{2n+1}(x)$ has constant sign from a certain n_0 onward (as $1 + A_{2n}(x) > 0$ for large n). Similarly, $A_{2n}(x)$ has constant sign from a certain n_1 onward. In either case it must be the positive sign, for otherwise

$$|A_{2n+1}(x)| > |A_{2n-1}(x)|,\ n > n_1,$$
or
$$|A_{2n+2}(x)| > |A_{2n}(x)|,\ n > n_0,$$

which implies that $\lim_{n\to\infty} A_{2n+1}(x) \neq 0$ or $\lim_{n\to\infty} A_{2n}(x) \neq 0$. Hence $A_k(x) > 0$ from a certain k_0 onward. It follows that

$$A_{k_0-1}(x) = A_{k_0+1}(x)(1 + A_{k_0}(x)) \geqslant 0 \quad \textbf{(6)}$$

and similarly $A_{k_0-2}(x) \geqslant 0, \ldots, A_1(x) = x \geqslant 0$.

Clearly $x \neq 0$.

4. We have:

(*a*) $A_{2n}(x)$ is continuous and decreasing in $x \geqslant 0$ $(n = 0, 1, \ldots)$.

(*b*) $A_{2n+1}(x)$ is continuous and increasing in $x \geqslant 0$ $(n = 0, 1, \ldots)$.

(*c*) $A_{2n}(x)$ is decreasing in $n = 0, 1, \ldots$ $(x \geqslant 0)$.

(*d*) $A_{2n+1}(x)$ is decreasing in $n = 0, 1, \ldots$ $(x \geqslant 0)$.

Furthermore, increasing and decreasing may be replaced by strictly increasing and strictly decreasing when $x > 0$ (except for $n = 0$ in (*a*)).

Proof. Parts (*c*) and (*d*) follow from step 1, (4) and (5). Then (*a*) and (*b*) follow from (4) and (5) by simultaneous induction on n.

5. Henceforth we assume that $x > 0$.

We have

$$\sum_{n=0}^{N} A_{2n+1}(x)A_{2n+2}(x) = 1 - A_{2N+2}(x),\quad N = 0, 1, \ldots, \quad \textbf{(7)}$$

and

$$\sum_{n=0}^{N} A_{2n+2}(x)A_{2n+3}(x) = x - A_{2N+3}(x),\quad N = 0, 1, \ldots. \quad \textbf{(8)}$$

Proof. These follow from summing the identity $A_{n+2}(x) - A_{n+1}(x)A_{n+2}(x) = A_n(x)$ over $n = 0, 2, \ldots, 2N$ and $n = 1, 3, \ldots, 2N + 1$, respectively, recalling that $A_0(x) = 1$ and $A_1(x) = x$.

6. If $\lim_{n\to\infty} A_n(x) = 0$, then

$$A_0(x) > A_1(x) > A_2(x) > \ldots. \quad \textbf{(9)}$$

Proof. We have by (7), (8), and the assumption

$$A_{2N+2}(x) = \sum_{n=N+1}^{\infty} A_{2n+1}(x)A_{2n+2}(x),$$
$$A_{2N+3}(x) = \sum_{n=N+1}^{\infty} A_{2n+2}(x)A_{2n+3}(x).$$

Hence, by these two formulas and 4(*c*),

$$A_{2N+1}(x) = \sum_{n=N+1}^{\infty} A_{2n}(x)A_{2n+1}(x) > \sum_{n=N+1}^{\infty} A_{2n+2}(x)A_{2n+1}(x) = A_{2N+2}(x)$$

and similarly

$$A_{2N}(x) = \sum_{n=N}^{\infty} A_{2n+1}(x)A_{2n+2}(x) > \sum_{n=N}^{\infty} A_{2n+3}(x)A_{2n+2}(x) = A_{2N+1}(x).$$

7. There is at most one x for which $\lim_{n\to\infty} A_n(x) = 0$.

Proof. Assume that $\lim_{n\to\infty} A_n(y) = \lim_{n\to\infty} A_n(x) = 0$ for $y > x$. Now

$$\frac{A_{2n+2}(y)}{A_{2n+2}(x)} = \frac{A_n(y)}{A_n(x)} \cdot \frac{1 + A_{n+1}(x)}{1 + A_{n+1}(y)}, \ n = 0, 1, \ldots$$

so that

$$\frac{A_{2n+2}(y)}{A_{2n+2}(x)} = \prod_{k=0}^{n} \frac{1 + A_{2k+1}(x)}{1 + A_{2k+1}(y)},$$

$$\frac{A_{2n+1}(y)}{A_{2n+1}(x)} = (y/x) \prod_{k=0}^{n} \frac{1 + A_{2k}(x)}{1 + A_{2k}(y)}.$$

It follows from 4(*b*) that $A_{2n+2}(y)/A_{2n+2}(x)$ decreases to a limit $L_1 < 1$, and it follows from 4(*a*) that $A_{2n+1}(y)/A_{2n+1}(x)$ increases to a limit $L_2 > 1$ (possibly ∞) as $n \to \infty$. Hence

$$\frac{A_{2n+2}(y)}{A_{2n+1}(y)} \cdot \frac{A_{2n+1}(x)}{A_{2n+2}(x)} \to (L_1/L_2) < 1.$$

By (9) and (1), however,

$$1 \geq \frac{A_{2n+2}(y)}{A_{2n+1}(y)} \geq \frac{A_{2n+3}(y)}{A_{2n+1}(y)} = \frac{1}{1 + A_{2n+2}(y)} \to 1,$$

and

$$1 \leq \frac{A_{2n+1}(x)}{A_{2n+2}(x)} \leq \frac{A_{2n}(x)}{A_{2n+2}(x)} = 1 + A_{2n+1}(x) \to 1$$

as $n \to \infty$, which is a contradiction.

8. There exists a unique x such that $\lim_{n\to\infty} A_n(x) = 0$.

Proof. It is clear from 4(*c*) and 4(*d*) that for $x \geq 0$, $a(x)$ and $b(x)$, as defined in (2), exist, and that $a(x)$ is increasing and $b(x)$ is decreasing. It is also clear from (3) that $a(x)b(x) = 0$ for all such x. We shall use the formulas

$$1 - a(x) = \sum_{n=0}^{\infty} A_{2n+1}(x)A_{2n+2}(x) \tag{10}$$

$$x - b(x) = \sum_{n=0}^{\infty} A_{2n+2}(x)A_{2n+3}(x) \tag{11}$$

which follow from (7) and (8) by letting $N \to \infty$. Since $A_{2n+3}(x) < A_{2n+1}(x)$ for $x > 0$ we see that $1 - a(x) > x - b(x)$ there, and hence $b(1) > 0$, $a(1) = 0$. We also have $a(0) = 1$, $b(0) = 0$. Let

$$x_0 = \sup\{x \in [0,1] \mid a(x) > 0\}.$$

From (6) we see that

$$x_0 = \inf\{x \in [0,1] \mid b(x) > 0\}$$

as well. We shall show that $a(x_0) = b(x_0) = 0$. We first observe that if either of the two series in (10) and (11) converges uniformly on a subinterval $[c,d]$ of $[0,1]$ so does the other. This follows from the inequality

$$(1/2)A_{2n+1}(x)A_{2n+2}(x) < A_{2n+2}(x)A_{2n+3}(x) < A_{2n+1}(x)A_{2n+2}(x).$$

Therefore, as the terms in both series are continuous and non-negative, continuity of either of the two functions a and b on $[c,d]$ implies continuity of the other on $[c,d]$. Since $a(x) = 0$ for $x > x_0$ and $b(x) = 0$ for $x < x_0$, it thus follows that a and b are continuous on all of $[0,1]$ except perhaps at x_0.

Suppose that $a(x_0) > 0$. Then $b(x_0) = 0$. Let $b_0 = \lim_{x \downarrow x_0} b(x)$, and suppose that $b_0 > 0$. Then

$$A_{2k}(x) = \prod_{j=1}^{k-1} (1 + A_{2j-1}(x))^{-1} < (1 + b_0)^{-(k-1)}, \quad x \in (x_0,1]$$

and

$$A_{2k+1}(x) = \prod_{j=1}^{k} (1 + A_{2j}(x))^{-1} < (1 + a(x_0))^{-k}, \quad x \in [0,x_0].$$

This implies that the series in (10) and (11) both converge uniformly, so that both a and b are continuous on $[0,1]$. This is a contradiction, and so $b_0 = 0$. This means that b, and therefore a, is continuous on

[0,1], however, which is itself a contradiction, and so $a(x_0) = 0$. Similarly, the assumption that $b(x_0) > 0$ leads to a contradiction. Hence $a(x_0) = b(x_0) = 0$, and we are done.

We conclude by giving a reasonably efficient method of computing x_0 numerically. This method is based on the following observation: For any $x \geq 0$, $x \neq x_0$, the sequence $A_0(x), A_1(x), A_2(x), \ldots$ is not monotonic (otherwise both $a(x)$ and $b(x)$ would be zero). Let n_0 be the first value of n such that $A_{n+1}(x) > A_n(x)$. Then $A_{n_0+2k+1}(x) > A_{n_0+2k}(x)$ for every integer $k \geq 0$, so $x < x_0$ or $x > x_0$ according to whether n_0 is odd or even. Thus we may approximate x_0 by using a bisection procedure, with the resulting value as stated.

86-4 by Andy Liu (University of Alberta, Canada)

One hundred mathematicians, all of different heights, line up in a 10-by-10 array. Each mathematician is shorter than all the others to the east or south of him. The shortest mathematician leaves the array, setting off a chain of "movements". Any vacant space is filled by either of the mathematicians immediately to the east or south of it, whoever is the shorter of the two. If there is only one mathematician for "movement", he moves. Finally, a new mathematician, taller than all the others, fills the vacant southeast corner of the array. This cycle of events then repeats. Earlier in the cycle in which mathematician "stretch" joined the array, the northwest corner was filled by the mathematician to the east of it. In the following cycle, mathematician "stretch" is sure to move. What is the probability that he moves northward?

Solution by the Proposer

Label the 100 positions in the array (i,j), $1 \leq i, j \leq 10$, where (1,1), (1,10), (10,10), and (10,1) are the NW, NE, SE, and SW corners, respectively.

Exactly 9 mathematicians move northward in each cycle. Let them be in positions $(2,a_1)$, $(3,a_2)$, $\ldots$, $(10,a_9)$ moving to $(1,a_1)$, $(2,a_2)$, $\ldots$, $(9,a_9)$, respectively, with $1 \leq a_1 \leq a_2 \leq \ldots \leq a_9 \leq 10$.

In the cycle in which mathematician "stretch" joins the array, we are given that $a_1 \geq 2$. After the move, the mathematician in $(1,a_1)$ is taller than the one in $(2,a_1 - 1)$ since the former just came from $(2,a_1)$, while the latter did not move. Similarly, the mathematicians in $(2,a_2)$, $\ldots$, $(9,a_9)$ are taller than those in $(3,a_2 - 1)$, $\ldots$, $(10,a_9 - 1)$, respectively.

In the next cycle, the mathematician now in $(1,a_1)$ cannot move: not out of the array because (s)he is not at the northwest corner, not northward because (s)he is at the northern border of the array, and not westward because the mathematician in $(2,a_1 - 1)$, being shorter, precludes that. Then none of the mathematicians in positions $(1,b)$, with $b \geq a_1$, can move either. Then, by a similar argument as for $(1,a_1)$, the mathematician in $(2,a_2)$ cannot move, nor can anyone in positions $(2,b)$, with $b \geq a_2$, move. Continuing in this way, we find that no one in positions $(9,b)$, with $b \geq a_9$, can move, so all mathematicians in positions $(10,b)$, with $b \geq a_9$, including "stretch" in (10,10), must move westward, and the desired probability is 0.

Problem 86-5* by Frank Schmidt (Bryn Mawr College, USA)

For each positive integer n, does there exist a polynomial $P(x)$ of degree n, not of the form $c(x - a)^n$, such that P and all its derivatives have integer roots?

Solution by Helmut Kloke (Universität Dortmund, FRG)

If $n = 2$, let $P(x) = x^2 - 2ax$, for any non-zero integer a. Otherwise, let $P_n(x) = x^n - nx^{n-1}$. Then $P_n(x)$ has all its roots integers and its derivative is $nP_{n-1}(x)$, so by induction $P_n(x)$ is a polynomial as desired.

Also solved by Attila Pethö (Universitatis Debreceniensis, Hungary) and Clark Carroll (St. John Fisher College, Rochester NY).

The DaVinci Sequence: Quickie 87-4† by the Column Editor

Find the last term of the following sequence: 11, 31, 71, 91, 32, 92, 13, 73, 14, 34, 74, 35, 95, 16, 76, 17, 37, 97, 38, 98, —.

Solution by the Proposer

Recall that Leonardo DaVinci kept his private notebooks in mirror writing. When one reads the numbers from right-to-left, rather than left-to-right, the solution appears instantaneously.